

Coherent Random Lasing and "Almost Localized" Photon Modes

V. M. Apalkov¹, M. E. Raikh¹, B. Shapiro²

¹*Department of Physics, University of Utah, Salt Lake City, UT 84112, USA*

²*Department of Physics, Technion-Israel Institute of Technology, Haifa 32000, Israel*

A pulse of light, injected into a weakly disordered dielectric medium, typically, will leave its initial location in a short time, by diffusion. However, due to some rare configurations of disorder, there is a possibility of formation of high quality resonators which can trap light for a long time. We present a rather detailed, quantitative study of such random resonators and of the "almost localized" states that they can support. After presenting a brief review of the earlier work on the subject, we concentrate on a detailed computation of the "prefactor": knowledge of the latter is crucial for verifying the viability of the random resonators and their areal density. Both short range disorder (white noise) and correlated disorder are studied, and the important effect of the correlation radius, R_c , on the probability of formation of resonators with a given quality factor Q is discussed. The random resonators are "self-formed", in the sense that no sharp features (like Mie scatterers or other "resonant entities") are introduced: our model is a featureless dielectric medium with fluctuating dielectric constant. We point out the relevance of the random resonators to the recently discovered phenomenon of coherent "random" lasing and review the existing work on that subject. We emphasize, however, that the random resonators exist already in the *passive* medium: gain is only needed to "make them visible".

I. INTRODUCTION

There exists a formal analogy between the Schrödinger equation describing electron motion in a random potential, and the scalar wave equation for light propagation in a medium with fluctuating dielectric constant. Consider, for concreteness, a two-dimensional geometry (quantum well for an electron and a thin film for light). Then both equations can be presented in the form

$$\Delta_{\rho}\psi + [k^2 - U(\rho)]\psi = 0, \quad (1)$$

where ρ is the in-plane coordinate. For an electron with mass m and energy E , moving in a random potential $V(\rho)$, the parameters k^2 and $U(\rho)$ are

$$k^2 = 2mE, \quad U(\rho) = 2mV(\rho). \quad (2)$$

For a light wave with frequency ω , traveling in a medium with dielectric constant ϵ , corresponding expressions for k^2 and $U(\rho)$ take the form

$$k^2 = \epsilon \left(\frac{\omega}{c}\right)^2, \quad U(\rho) = -\delta\epsilon(\rho) \left(\frac{\omega}{c}\right)^2, \quad (3)$$

where $\delta\epsilon(\rho)$ is the fluctuating part of the dielectric constant.

The analogy between electrons and light is incomplete. For one thing, the "scattering potential" for light depends on frequency and it vanishes in the limit of zero frequency. Moreover, in a dielectric medium (positive ϵ) k^2 must be positive which implies that there can be no bound states for light, as long as $\delta\epsilon(\rho)$ is assumed to vanish outside some finite region in space. Therefore the simple notion of a binding potential well does not exist for light: for instance, a dielectric sphere embedded in a uniform medium cannot bind photons, regardless of whether its dielectric constant is larger or smaller than that of the surrounding medium. Thus, unlike electrons, photons will always escape from a dielectric medium into the surrounding air. However, under appropriate conditions, they can be trapped within the sample for a long time. The main purpose of this paper is to discuss trapping of light in a weakly disordered dielectric film.

The random term in Eq. (1) is zero on average; its statistical properties are described by the r.m.s. value, U_0 , and the correlator $K(\rho_1 - \rho_2)$

$$\langle U(\rho) \rangle = 0, \quad \langle U(\rho_1)U(\rho_2) \rangle = U_0^2 K(\rho_1 - \rho_2), \quad K(0) = 1. \quad (4)$$

It is known [1] that in two dimensions the nature of solutions of Eq. (1) is governed by the dimensionless conductance kl , where l is the transport mean free path. Using the golden rule, the product kl can be expressed through the correlator, K , as follows

$$\begin{aligned}
(kl)^{-1} &= \frac{U_0^2}{4\pi k^4} \int dq d\phi q^3 \delta(q^2 + 2kq \cos \phi) \int d^2\rho K(\rho) \exp(iq\rho) = \\
&= \frac{2U_0^2}{k^2} \int_0^{\pi/2} d\alpha \sin^2 \alpha \int_0^\infty d\rho \rho K(\rho) J_0(2k\rho \sin \alpha),
\end{aligned} \tag{5}$$

where J_0 is the Bessel function of zero order. The above integral can be evaluated analytically if we choose a gaussian form for the correlator $K(\rho) = \exp(-\rho^2/R_c^2)$. Substituting this form into (5), we obtain

$$(kl)^{-1} = \pi \left(\frac{U_0 R_c}{2k} \right)^2 F \left(\frac{k^2 R_c^2}{2} \right), \tag{6}$$

where the dimensionless function F is defined as

$$F(x) = e^{-x} [I_0(x) - I_1(x)]. \tag{7}$$

Here I_0 and I_1 are the modified Bessel functions of zero and first order, respectively.

As a simple example of a realistic two-dimensional disorder, consider a system of disks with random positions of the centers and with fluctuating radii described by a normalized distribution function, $\Phi(R)$. Within a disk, we have $U(\boldsymbol{\rho}) = U_d$, whereas outside the disk $U(\boldsymbol{\rho}) = 0$. If the filling fraction, f , is low, $f \ll 1$, the positions of the centers of the disks are uncorrelated. Then $K(\rho)$ is determined by the overlap between two circles of the same radius with centers shifted by ρ . A straightforward calculation yields

$$\begin{aligned}
\langle U(\boldsymbol{\rho}_1) U(\boldsymbol{\rho}_1 + \boldsymbol{\rho}) \rangle &= \frac{2U_d^2 f}{\pi} \int dR \theta\left(R - \frac{\rho}{2}\right) \Phi(R) \\
&\times \left\{ \arcsin \sqrt{1 - \left(\frac{\rho}{2R}\right)^2} - \frac{\rho}{2R} \sqrt{1 - \left(\frac{\rho}{2R}\right)^2} \right\},
\end{aligned} \tag{8}$$

where $\theta(x)$ is the step-function. The actual distribution function, $\Phi(R)$, is governed by various technological factors. However, it is reasonable to assume that small values of R are strongly unlikely, and that $\Phi(R)$ falls off abruptly at large R . Consider as an example the distribution $\Phi(x) = c^3 x^4 \exp(-cx^2)$, where $x = R/\bar{R}$ (\bar{R} is the average radius) and $c = 64/9\pi$. The distribution is designed so as to yield the value 50% for the relative spread in R , i.e., $\langle |R - \bar{R}| \rangle / \bar{R} = 0.5$. The result of calculation of the integral (8) using this distribution is shown in Fig. 1 together with its gaussian fit, which yields $U_0^2 = 0.86 U_d^2 f$ and $K(\rho) = \exp(-3.4\rho^2/\bar{R}^2)$. From this fit we conclude that the gaussian form of the correlator, $K(\rho)$, corresponds to a quite generic distribution of R .

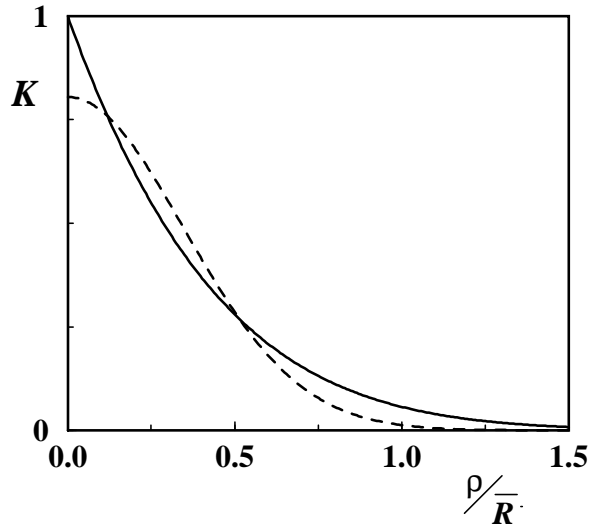


FIG. 1. Correlation function (solid line) for the model of randomly distributed disks $U(\rho) = U_d \theta(R - \rho)$ with fluctuating radii is shown together with its gaussian fit (dashed line).

Equation (6) provides the explanation why the criterion for strong localization, $kl < 1$, can be easily satisfied for electrons, but very hard to achieve in the case of photons. Indeed, for electrons Eq. (6) can be presented as

$$(kl)^{-1} = \left(\frac{\pi m}{2E}\right) V_0^2 R_c^2 F(mER_c^2), \quad (9)$$

while for photons Eq. (6) takes the form

$$(kl)^{-1} = \left(\frac{\pi}{4\epsilon}\right) \left(\frac{\omega}{c}\right)^2 \Delta^2 R_c^2 F\left(\frac{\epsilon\omega^2 R_c^2}{2c^2}\right), \quad (10)$$

where V_0 and Δ are the r.m.s. fluctuations of the potential and dielectric constant, respectively. It is seen from Eq. (9) that, since the function F is always smaller than one, the electrons are strongly localized in the *low-energy* domain $E < E_c$. The value of E_c is different for the short-range, $R_c \ll (mV_0)^{-1/2}$, and for the smooth, $R_c \gg (mV_0)^{-1/2}$, potentials. For short-range potential we can use the asymptotics $F(x)|_{x \ll 1} \approx 1$, which leads us to the estimate $E_c \sim mV_0^2 R_c^2$. Hence, $mE_c R_c^2 \sim (mV_0 R_c^2)^2 \ll 1$, so that the argument of F is indeed small at $E = E_c$. In the case of the smooth potential, the condition $mV_0 R_c^2 \gg 1$ suggests that the semiclassical description applies, so that $E_c \sim V_0$. Calculation of the mean free path based on the golden rule is inadequate in this case. To see this, note that at $E \sim V_0$, the argument of F in Eq. (9) is large, so that we can use the asymptotics $F(x)|_{x \gg 1} \propto x^{-3/2}$. Then Eq. (9) yields $l \sim R_c$ for $E \sim V_0$. Thus, for smaller E , namely $E < V_0$, we have $l < R_c$. The latter relation indicates the failure of the perturbation theory for $E < V_0$.

Now let us perform the similar analysis for photons. Using the small- x and large- x asymptotics of the function F , we obtain from Eq. (10)

$$kl|_{k_0 R_c \ll 1} = \frac{4\epsilon}{\pi(k_0 R_c \Delta)^2}, \quad kl|_{k_0 R_c \gg 1} = \frac{4\epsilon^{5/2} k_0 R_c}{\pi^{1/2} \Delta^2}, \quad (11)$$

where $k_0 = \omega/c$. Eq. (11) suggests that when the r.m.s. fluctuation Δ is weak, $\Delta \ll \epsilon$, we have $kl \gg 1$ *both* in the low-frequency ($k_0 R_c \ll 1$) and in the high-frequency ($k_0 R_c \gg 1$) domains. This peculiar result is due to the already mentioned frequency dependence of the "optical potential".

The above analysis, however, does not rule out the possibility of light localization in the *strongly scattering* media. In fact, first experimental indications of localization effects for microwaves were reported more than a decade ago [2,3]. In these experiments localization was inferred from the measurements of various transmission characteristics of the microwaves through the tube filled with a random mixture of aluminum and Teflon spheres. For optical frequencies [4,5,6], the strongly scattering medium used in first experiments, aimed at light localization, was a semiconductor (*GaAs*) powder. Measurements of the transmission vs. the sample thickness were complimented with measurements of the coherent backscattering (CBS) cone [7,8]. In the latter measurements localization manifests itself through the rounding of the CBS cone by limiting the maximal length of coherent path [9].

The early reports of the observation of wave localization both for microwaves and for light [2-4] were inconclusive because of the possibility that the results were affected by absorption. To get rid of this ambiguity, in the later experiments the CBS measurements [10] were performed on the macroporous *GaP* networks, which scatter light stronger than a powder [4]. This allowed the authors to rule out the absorption or the finite sample size as a source of rounding of the CBS cone. For microwaves [11-13], the recent progress in detecting localization is due to a novel approach to the analysis of the transmission data based on analysis of the relative size of the transmission fluctuations. This approach permits one to detect localization even in the presence of absorption.

In the weakly scattering *active* medium the propagation of light remains diffusive. However, the interplay of the diffusion and the gain-induced amplification can be very nontrivial. Namely, this interplay can give rise to the *incoherent* random lasing, predicted by V. I. Letokhov [14]. As it was pointed out in Ref. [14], there is a close analogy between multiplication of neutrons in course of the chain reaction and photons in the amplifying disordered medium (photonic bomb). Upon first experimental observation of incoherent random lasing [15], it was subsequently reproduced for various realizations of the gain media and different types of disorder. Comparison of theoretical [16-21] and experimental results [15,22-42] has confirmed that the diffusion theory, which neglects the interference effects, is quite sufficient for the description of *incoherent* lasing. Except for studies on powder grains of laser crystal materials [37-40] and π -conjugated polymer films [41,42], the majority of experiments [15,22-36] have used dye solutions as amplifying media. Colloidal particles suspended in a solution served as random scatterers. These scatterers are responsible for *nonresonant feedback* required for incoherent lasing. The essence of this feedback is that the light amplification length, \tilde{l}_a (in the absence of disorder), is significantly shortened [to $\sim (\tilde{l}_a)^{1/2}$] in the disordered

medium, when the light propagation is diffusive. The threshold condition for incoherent lasing corresponds to the gain magnitude at which $(l\bar{l}_a)^{1/2}$ becomes of the order of the sample size.

In contrast to incoherent lasing, the recent discovery of *coherent* random lasing [43-46] adds a new dimension to the physics of light propagation in disordered media. Coherent random lasing emerges as the degree of disorder increases, so that the mean free path, l , becomes progressively smaller. The fact that the light, emitted from a disordered sample, is truly coherent, which does not necessarily follow [47,48] from drastic narrowing of the emission spectrum, observed in Refs. [43-46], was later convincingly demonstrated in photon statistics experiments [49,50]. In the absence of mirrors, it is evident that, in order to support the coherent lasing, the disordered medium *itself* should assume their role. The latter is feasible only due to the *interference effects*, that are not captured within the diffusion picture. More quantitatively, in order for the random medium to play the role of a Fabry-Perot resonator, it is necessary that certain eigenfunctions of Eq. (1) were either completely localized or *almost* localized. Almost localized solution can be, roughly, envisaged as a very high *local* maximum of the extended eigenfunction $\psi(\rho)$ of Eq. (1). If this maximum is viewed as a *core* of $\psi(\rho)$, then the delocalized *tail* (see Fig. 2) can be viewed as a source of *leakage*. In other words, the core itself, being not an exact eigenfunction, can be viewed as a solution of Eq. (1) corresponding to a *complex* eigenvalue $\text{Im } k^2 \neq 0$. Then the weak leakage translates into a small value of the imaginary part of k^2 . Inverse of this imaginary part determines the *lifetime* of the core. The higher is the local maximum of $\psi(\rho)$, the longer is the lifetime. Making link to the Fabry-Perot resonator, the ratio $k^2/\text{Im } k^2$ can be identified with a quality factor, Q . Thus, from the perspective of the coherent random lasing, the right question to be asked is: how high are the attainable quality factors of the almost localized solutions of Eq. (1) at a given parameters of disorder, i.e., magnitude, Δ , and correlation radius, R_c .

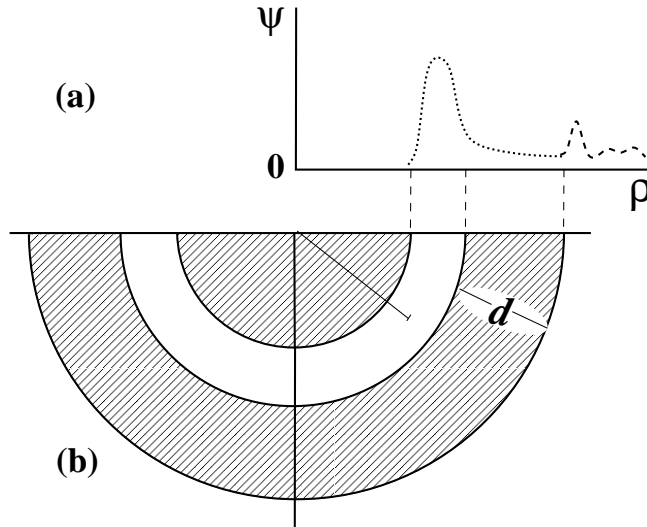


FIG. 2. (a) Spatial distribution of the wave function of the anomalously localized solution, $\psi(\rho)$, of Eq. (1) is shown schematically. (b) Spatial distribution of the dielectric constant, $\epsilon(\rho)$, corresponding to the trap, responsible for the solution $\psi(\rho)$; $\epsilon(\rho) = \epsilon$ outside the blank region. Only the lower half of the trap is shown.

We address this question in the present paper. However, prior to discussing the almost localized solutions of Eq. (1), a subtle point must be clarified. Namely, Eq. (1) with the “potential” being real, describes propagation of light in a *passive* medium. It might be argued that, in the presence of gain, which is required for lasing, the spatial structure of the solutions of Eq. (1) undergoes a drastic change, so that the almost localized modes of a passive random medium and actual *lasing* modes have little in common. In fact, it is well known both from theory [51] and from CBS experiments [52] that the *diffusive* trajectories of light within a random medium get *elongated* in the presence of the gain. With regard to coherent random lasing, it was initially claimed [53] that the gain *facilitates localization* of light. This claim was even supported by the numerical simulations [53]. However, in the later theoretical [54-56] and experimental [57] papers it was explicitly stated that, similarly to the conventional lasers with Fabry-Perot resonators, the gain only *reveals* high- Q solutions of Eq. (1) existing in the passive medium. Thus, in the present paper, we will focus exclusively on the passive disordered media. The question about the likelihood of formation of the disorder-induced resonators is central to the understanding of the coherent random lasing. This question is at the core of the ongoing in-depth experimental studies [57-65]. Except for Ref. [66], theoretical papers on random lasing [56,67-71] do not address this question.

Let us finally mention that a similar question of trapping electrons, for a long time, in a weakly disordered conductor

was addressed long ago in the context of electronic transport [72-75]. However, the "almost localized" states discussed in [66,76,77] and, further, in this paper are quite different from the "prelocalized" states studied in [72-75]. Wave functions of the "almost localized" states are confined primarily to a small ring (a high- Q resonator). Moreover, these states are extremely sensitive to the value of the correlation radius, R_c , of the disordered potential. We shall see below that, for a given parameter $kl > 1$, the likelihood of high- Q "almost localized" modes is very low for the short-range disorder but sharply increases with R_c . Such features are absent for the "prelocalized" states [72-75], with their comparatively large spatial extent.

II. LIKELIHOOD OF RANDOM RESONATORS IN A DISORDERED FILM

A. Intuitive Scenario of Random Cavities Based on Recurrent Scattering

The first experimental work on coherent random lasing by Cao *et. al.* [43] was carried out on thin (with a width $\sim 2\pi/k_0$) zinc oxide (ZnO) polycrystalline films. Laser action manifested itself through a drastic narrowing of the emission spectrum when the optical excitation power exceeded a certain threshold. The authors of Ref. [43] realized at the time that coherent lasing requires a resonator (cavity). They conjectured that, in the absence of traditional well-defined resonator (as in a semiconductor laser), the cavities in polycrystalline films are "self-formed" due to strong optical scattering. For a microscopic scenario of the formation of such cavities, they alluded to the remark made in Ref. [18] that closed-loop paths of light can serve as "random ring cavities". The importance of these closed-loop paths (recurrent scattering events) was pointed out earlier [78], when they were invoked for the explanation of the magnitude of the CBS albedo. With regard to CBS, the effect of recurrent scattering events, e.g., the events in which the *first* scattering (of the incident wave) and the *last* scattering (of the outgoing wave) are provided by the *same* scatterer, is that these events *do not* contribute to the CBS, thus reducing the albedo in the backscattering direction.

The fact that recurrent events show up in CBS certainly does not allow to automatically identify closed-loop paths with resonator cavities. Therefore, the feasibility of the scenario of random cavities, adopted in Refs. [43,44,58,59] for interpretation of experimental results, was later put in question [79]. The arguments against this scenario were the following. Since in each scattering act most of the energy gets scattered out of the loop, an unrealistically high gain would be required to achieve the lasing threshold condition for such a loop. Also the loops of scatterers are likely to generate a broad frequency spectrum rather than isolated resonances.

Certainly the picture of random cavities representing a certain spatial arrangement of *isolated* scatterers is too naive. This, however, does not rule out the entire concept of disorder-induced resonators. Although sparse, the disorder configurations that trap the light for long enough time can be found in a sample of a large enough size, and a single such configuration is already sufficient for lasing to occur. Therefore, under the condition $kl \gg 1$, which implies that *overall* scattering is weak, the conclusion about the relevance of random cavities can be drawn only upon the *calculation* of their likelihood. This calculation is described in the rest of this Section first on qualitative, and then on quantitative levels.

Making link to the discussion in the Introduction: by pursuing the random cavity scenario Cao *et. al.* have intuitively arrived to the concept of prelocalized states, that was introduced in transport more than a decade ago. The fundamental difference between the prelocalized states and recurrent events is that recurrent events emerge in the higher order [in parameter kl^{-1}] of the *perturbation theory* (they correspond to the specific type of diagrams called Hikami-boxes [80]), whereas the formation of prelocalized states is a genuinely *nonperturbative* effect. Already in the first analytical approach to the problem of prelocalized states [81], it was demonstrated that, in order to capture an anomalously slow tail of the conductance relaxation (which is due to trapping), *all* the orders of the perturbation theory must be taken into account.

B. Optimal Fluctuation Approach to the Problem

1. Qualitative Discussion.

Similarly to the treatment in [43,53,59] we restrict our consideration to the two-dimensional case (a disordered film). Regarding the geometry of a random resonator, we adopt the idea proposed by Karpov for trapping acoustic waves [76] and electrons [77] in three dimensions.

Suppose that within a certain stripe the effective in-plane dielectric constant of a film is enhanced by some small value $\epsilon_1 \ll \epsilon$. Then such a stripe can play a role of a *waveguide*, i.e., it can capture a transverse mode, as it is illustrated in Fig. 3. There is no threshold for such a waveguiding, which means that the transverse mode will be

captured even if the width of the stripe is small. Now, in order to form a resonator, one has to roll the stripe into a ring. Upon this procedure, the mode propagating along the waveguide transforms into a whispering-gallery mode of a ring. An immediate consequence of the curving of the waveguide is emergence of the evanescent leakage - the optical analog of the under-the-barrier tunneling in quantum mechanics (see Fig. 2). This leakage is responsible for a finite lifetime of the whispering-gallery mode. Thus we have specified the structure of the weakly decaying solutions of Eq. (1), discussed in the Introduction. Namely, the mode of the waveguide plays the role of the *core*, while delocalized tail (see Fig. 2) reflects the evanescent leakage. Due to the azimuthal symmetry, the modes of the resonator are characterized by the angular momentum, m . Denote by $\mathbf{N}_m(kl, Q)$ the areal density of resonators with quality factor Q in the film with a transport mean free path l . Obviously, in the diffusive regime, $kl > 1$, the density $\mathbf{N}_m(kl, Q)$ is exponentially small for $Q \gg 1$. In this domain $\mathbf{N}_m(kl, Q)$ can be presented as

$$\mathbf{N}_m(kl, Q) \propto e^{-S_m(kl, Q)}. \quad (12)$$

Let us first give a qualitative estimate for S_m , which reveals its sensitivity to the strength and the range of the disorder (Δ^2 and R_c). Since m is the number of wave lengths along the ring, its radius is $\rho_0 = m/k$. The ring waveguide can support a weakly decaying mode only if its width w satisfies the condition $w(\epsilon_1/\epsilon)^{1/2} > k^{-1}$. A straightforward estimate for the decay time due to evanescent leakage, i.e., the quality factor Q of the waveguide results in $\ln Q \sim k\rho_0(\epsilon_1/\epsilon)^{3/2}$. Since the number $k\rho_0 = m$ is large, a relatively small fluctuation of the dielectric constant within the area $2\pi\rho_0 w$ of the ring can produce a large value of Q .

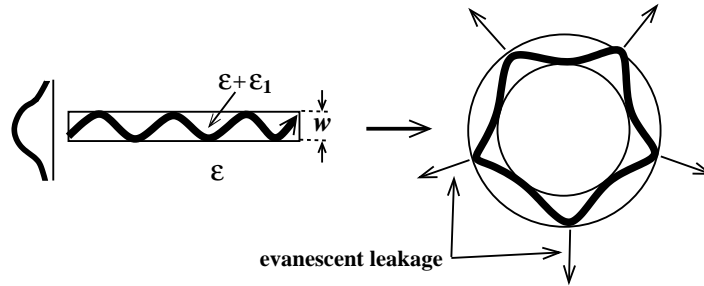


FIG. 3. Rationale for the structure of the resonator. Upon wrapping a stripe with enhanced dielectric constant into a ring, a waveguided mode transforms into a whispering-gallery mode.

The probability \mathcal{W} for creating the required fluctuation strongly depends on R_c . For a short range disorder ($k_0 R_c \ll 1$) fluctuations of order ϵ_1 should occur *independently* in a large number, $N \sim \rho_0 w / R_c^2$, of spots within the ring, so that the probability $\mathcal{W} \sim \exp(-N\epsilon_1^2/\Delta^2)$. In the other extreme of strongly correlated disorder, when $R_c \gg w$, the number of independent spots is much smaller, $N' \sim \rho_0 / R_c$ (the number of squares with a size R_c needed to cover the ring). Correspondingly, the probability $\mathcal{W} \sim \exp(-N'\epsilon_1^2/\Delta^2)$ is much larger than for the short range case. Finally, using the relation between ϵ_1 and Q , the probability $\mathcal{W} = \exp(-S_m)$ can be rewritten in terms of Q , thus, yielding an estimate for S_m . For the short range case ($k_0 R_c \ll 1$) we obtain $S_m \sim kl \ln Q$, where the mean free path l is proportional to $(R_c \Delta)^{-2}$ [see Eq. (11)]. In the opposite limit of a smooth disorder Eq. (11) yields for the transport mean free path $l \sim R_c / \Delta^2$. Then we have $S_m \sim N'\epsilon_1^2/\Delta^2 \sim l(\ln Q)^{4/3}/(kR_c^2 m^{1/3})$. Thus, for given kl and Q , the density of resonators is the higher the smoother is the disorder. This conclusion is central to our study and will be addressed below in more detail.

2. Quantitative Results from the Optimal Fluctuation Approach.

The above program can be carried out analytically [66] with the use of the optimal fluctuation approach [82,83]. This approach is based on the idea that, when the exponent, S_m , in $\mathbf{N}_m(kl, Q)$ is large, then the major contribution to $\mathbf{N}_m(kl, Q)$ comes from a certain specific disorder realization. In application to random resonators, the optimal fluctuation procedure reduces to finding the *most probable* fluctuation of the dielectric constant which is able to trap the light for a long time $\sim \omega^{-1}Q$. Assuming that the fluctuation is azimuthally symmetric (see Fig. 2), the shape of the optimal fluctuation can be found explicitly [66] yielding the following expression for exponent S_m

$$S_m = 2^4 3^{-3/2} \pi^{1/2} m \left(\frac{\epsilon_1^3}{\epsilon} \right)^{1/2} \frac{\Phi(\epsilon_1^{1/2} k_0 R_c)}{(\Delta k_0 R_c)^2}, \quad (13)$$

where $\epsilon_1 = \epsilon(3 \ln Q / 2m)^{2/3}$. The analytical expression for the function $\Phi(u)$ is the following

$$\Phi(u) = \frac{3^{3/2}}{2^6} \frac{(5 + \sqrt{9 + 16u^2})^{5/2}}{(3 + \sqrt{9 + 16u^2})^{3/2}}. \quad (14)$$

Recall now, that we are interested in the density of random resonators at a *given* value of kl . The transition from Δ to l is accomplished by using Eq. (11). For the short range case, when $R_c \rightarrow 0$ and $\Phi(u) \rightarrow 1$, we obtain

$$S_m(k_0 R_c \ll 1) = 2 \left(\frac{\pi^3}{3} \right)^{1/2} kl \ln Q. \quad (15)$$

To trace the change of S_m with increasing R_c it is convenient, after using Eq. (11), to present Eq. (13) in the form

$$\frac{S_m(k_0 R_c > 1)}{S_m(k_0 R_c \ll 1)} = \frac{\Phi(\epsilon_1^{1/2} k_0 R_c)}{\pi^{1/2} (\epsilon_1^{1/2} k_0 R_c)^3}. \quad (16)$$

It is seen from Eq. (16) that S_m falls off rapidly with increasing R_c . In the domain $k_0 R_c > 1$, but $\epsilon_1^{1/2} k_0 R_c < 1$ we have $\Phi \approx 1$, so that $S_m \propto (k_0 R_c)^{-3}$. For larger R_c we have $\Phi(u) \propto u$. In this domain S_m decreases slower with R_c : $S_m \propto (k_0 R_c)^{-2}$

$$S_m(k_0 R_c \gg 1) = \frac{3^{4/3} \pi}{4^{5/3}} \frac{kl \ln^{4/3} Q}{m^{1/3} (k R_c)^2}. \quad (17)$$

Asymptotic expressions (15) and (17) agree with the results of the qualitative derivation, with all numerical factors now being determined. We emphasize that Eqs. (15) and (17) apply *for a given* kl value, so that the decrease of S_m with R_c *leaves the backscattering cone unchanged*.

3. Estimates.

Equation (15) quantifies the effectiveness of trapping of light in a random medium with point-like scatterers. It follows from Eq. (15) that the likelihood of high- Q cavity is really small. Indeed, even for rather strong disorder, $kl = 5$, the exponent, S_m , in the probability of having a cavity with a quality factor $Q = 50$ is close to $S_m = 120$. We emphasize that in two dimensional case under consideration, this exponent does not depend on m and, thus, on the cavity radius $\rho_0 = m/\epsilon^{1/2} k_0$. More accurate calculation [84], taking into account the corrections to Eq. (15), indicates that S_m as a function of m has a minimum at $m \sim (kl \ln Q)^{1/2}$.

To estimate the degree to which finite size of scatterers ($\sim R_c$) improves the situation, we choose $k_0 R_c \approx 2$, which already corresponds to the limit $k_0 R_c \gg 1$ in Eq. (11), but still allows to set $\Phi = 1$. Then for $Q = 50$, $kl = 5$ we obtain $S_m \approx 1.1$, suggesting that the resonators with this Q are quite frequent. In the latter estimate we have set $\epsilon = 4$.

C. Frequently Asked Questions

1. Why Rings?

The answer to this question is illustrated in Fig. 4. The distinguishing property of a ring is that the local curvature radius is the same at each point. Upon any deviation from the ring geometry, the curvature in a certain region of the fluctuation would be higher than in all other regions. Since the evanescent losses are governed by this curvature, the quality factor of the resonator would be determined exclusively by this region (see Fig. 4), so that the remaining low-curvature part would be “unnecessary”, in the sense, that a ring with a radius corresponding to the maximal curvature would have the same quality factor as a square in Fig. 4 but significantly higher probability of formation. It is also quite obvious that, for the purpose of supporting a wave-guided mode of the whispering-gallery type, a ring is much superior to a disk of the same radius: indeed, the internal area of the disk remains unused in the guiding process, whereas a heavy penalty in terms of probability is paid in creating this area.

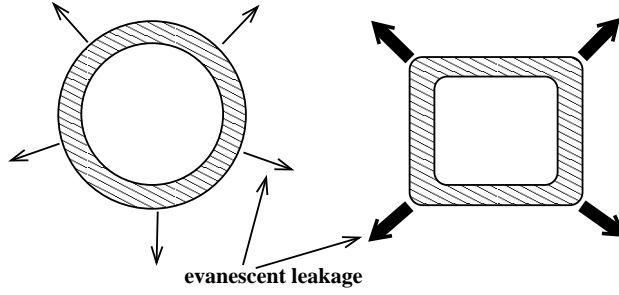


FIG. 4. This drawing illustrates the optimal character of the ring-shape resonator.

2. Why Smooth Disorder Facilitates Trapping?

At the qualitative level, the enhancement of the probability of formation of the cavity with increasing R_c can be understood for a toy model of the disorder, illustrated in Fig. 5. Suppose that all the disks, that model the scatterers, are identical. Then R_c scales with the radius of the disk, R . Since the disks cannot interpenetrate, the ring-shaped cavity corresponds to their arrangement in the form of a necklace. The probability of formation of such a cavity can be estimated as follows. Suppose that a sector, $\delta\phi$, is “allocated” for a single disk. The probability to find a disk within this sector, at the distance ρ_0 from the center, is $\sim n(\rho_0\delta\phi)^2$, where n is the concentration of the disks. Thus, the probability of formation of the necklace is $\exp\left[-\frac{2\pi}{\delta\phi} \ln\left(\frac{1}{n\rho_0^2(\delta\phi)^2}\right)\right]$, where $\frac{2\pi}{\delta\phi}$ is the number of sectors. It is obvious that if a necklace is “loose”, the quality factor of the corresponding cavity would be low. In order for Q to be high, neighboring disks must almost touch each other. This implies that $\delta\phi \approx 2R/\rho_0$. Then the above estimate for probability takes the form $\exp\left[-\frac{\pi\rho_0}{R} \ln\left(\frac{1}{f}\right)\right]$, where $f = n\pi R^2$ is the filling fraction. This probability increases exponentially with R , i.e., with R_c , reflecting the fact that, for a given ρ_0 , the number of disks to be arranged is smaller when R is larger. The above estimate was based on the assumption that the positions of the disks are uncorrelated, i.e., $f \ll 1$ (in contrast to [54] where $f = 0.4$). We have used the model of *hard* disks as an easiest illustration of the role of R_c . Obviously, Eq. (13) does not apply to this model, since it was derived under the assumption that the statistics of the fluctuations is gaussian.

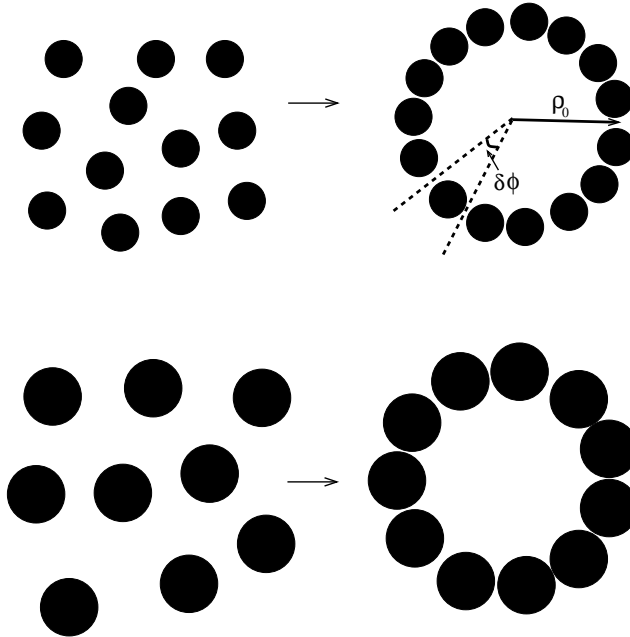


FIG. 5. A schematic illustration explaining why larger correlation radius for a fixed filling fraction facilitates trapping. A sector, shown with dashed lines, illustrates the tolerance in the arrangement of disks into a necklace.

3. “Vulnerability” of the Ring-Shaped Cavities.

The value S_m given by Eq. (13), which was derived within the optimal fluctuation approach, is the exponent in the probability of formation of an *ideal* ring. Obviously, any actual disorder realization is not ideal, in the sense, that actual distribution of dielectric constant differs from the optimal. For the same reason, the probability of formation of ideal necklace of the type shown in Fig. 5 is zero. In order for the probability to be finite, we should allow a certain *tolerance* in the positions of the centers of the disks, as illustrated in Fig. 5. In the conventional applications of the optimal fluctuation approach [85], deviations from the optimal distribution do not affect the value of the exponent, S_m . However, in application to the random cavities, we have searched for the fluctuation which is optimal *for a given trapping time*, $\omega^{-1}Q$. In this particular application, a “normal” gaussian deviation from the optimal geometry can have a catastrophic effect on trapping by scattering the light wave out of the whispering-gallery trajectory. This scattering is discussed below.

Scattering within the plane. Two-dimensional picture adopted throughout this paper, implies that electromagnetic field is confined within a thin film in the z -direction. This confinement results from the fact that the average dielectric constant of the film is higher than in the adjacent regions. Then the field distribution, $\mathcal{E}_0(z)$, along z corresponds to a transverse waveguide mode. For a given frequency, ω , the almost localized state on a ring can be destroyed due to the scattering into states with the *same* distribution of the field in the z -direction, which propagate freely along the film. More precisely, the almost localized state with a given angular momentum m , which is protected from the outside world by the centrifugal barrier, can be scattered out to the continua of states with smaller m ’s, for which there is no barrier. It is essential to estimate the lifetime, τ , with respect to these scattering processes and to verify that it is feasible to have τ larger than the prescribed trapping time, $\omega^{-1}Q$, so that the almost localized state is not destroyed. A rigorous treatment of this “scattering out” effect is quite involved and is done in Section III, for a Gaussian potential.

The effect can be also illustrated with the model of randomly positioned hard disks (Fig. 5), although the disorder in this model is non-gaussian. It is seen from Fig. 5, that spacings between the rings, which are due to tolerance, open a channel for the light escape, that is different from evanescent leakage. A *typical* lifetime with respect to such an escape is quite short, i.e., even a small tolerance, which affects weakly the exponent in the probability of the cavity formation, seems to be detrimental for trapping. At this point we emphasize that, in calculation of the scattering rate out of the whispering-gallery trajectory, the disks constituting the necklace must be considered as a *single entity*. As a result, for a *given* configuration of the disks, the rate of scattering out caused by a *single* disk must be multiplied by the following *form-factor*

$$\mathcal{F} = \int \frac{d\varphi_k}{2\pi} \left| \sum_i \exp(i\mathbf{k}\boldsymbol{\rho}_i) \right|^2 = \sum_{i,j} J_0(k|\boldsymbol{\rho}_i - \boldsymbol{\rho}_j|), \quad (18)$$

where $\boldsymbol{\rho}_i$ is the position of the center of the i -th disk in the necklace. The form-factor, \mathcal{F} , is the sum of $(\frac{\pi\rho_0}{R})^2 \gg 1$ terms. Out of this number, $(kR)^{-1} (\frac{\pi\rho_0}{R})$ terms (for $kR < 1$) and $\frac{\pi\rho_0}{R}$ terms (for $kR > 1$), for which $k|\boldsymbol{\rho}_i - \boldsymbol{\rho}_j| < 1$, are close to unity. The portion of these terms is small. Other terms have random sign. This leads us to the important conclusion that, for certain realizations of the necklace in Fig. 5, the form-factor can take anomalously small values. For these realizations the quality factor will be still determined by the evanescent leakage. The “phase volume” of these realizations is exponentially small and depends strongly on the model of the disorder.

Scattering out of the plane. Compared to the previous case, two modifications are in order. Firstly, since the final state of the scattered cavity mode is a plane wave with the wave vector pointing in a certain direction within the solid angle 4π , the expression Eq. (18) for the form-factor should be replaced by

$$\tilde{\mathcal{F}} = \sum_{i,j} \frac{\sin(k_0|\boldsymbol{\rho}_i - \boldsymbol{\rho}_j|)}{k_0|\boldsymbol{\rho}_i - \boldsymbol{\rho}_j|}. \quad (19)$$

Secondly, for $kR > 1$, i.e., when the disorder is smooth, scattering out of the plane that is caused by a single disk, requires a large wave vector transfer, $\sim k$. Thus, the corresponding rate is suppressed as compared to the in-plane scattering.

III. PREFACTOR

A. Qualitative discussion

In Sec. II we calculated the *probability* of the formation of a trap that captures light for a long time $\tau = \omega^{-1}Q$. However, the relevant characteristics of traps is their *areal density*. Within the optimal fluctuation approach, the relation between the probability and the areal density emerges in course of calculation of the prefactor [85,86]. Namely, the combination with dimensionality of the inverse area comes from the so called “zero modes”, which reflect the fact that the fluctuation can be shifted *as a whole* in both x and y directions. A typical shift, making two fluctuations independent, is of the order of the extent of the fluctuation in each direction. This suggests that the proportionality coefficient between the density of traps and the probability of the formation of the trap is roughly the inverse area of the fluctuation [85]. In our particular case, when the fluctuation is ring-shaped, the estimate for the *dimensional* prefactor is $\sim w^{-2}$, where w is the width of the ring (see Sec. II). The dimensionless part of the prefactor within the optimal fluctuation approach reflects the “phase volume” of the fluctuations, which perturb the shape of the optimal fluctuation leaving the “energy” k^2 unchanged.

For the almost localized modes, considered above, the situation with prefactor is qualitatively different from the case of the truly localized states, for which the optimal fluctuation approach was devised [82]. The specifics of the almost localized states is that their “energy”, k^2 , [see Eq. (1)] is degenerate with continuum of the propagating modes. As a result, a typical small perturbation of the shape of the ring will not only shift k^2 , but also cause the coupling of the trapped mode to the continuum, or in other words, the *additional leakage* will emerge due to the fluctuations. In order to incorporate this effect into the theory, the density of traps should be multiplied by the probability, $P(\tau)$, that the lifetime with respect to this additional leakage is longer than τ [84]. To estimate this probability, we consider the additional leakage for a *given* disorder realization

$$\frac{1}{\tau_U} = 2\pi \left(\frac{c}{k}\right) \sum_{\mu} \left| \int d\boldsymbol{\rho} \psi_0^*(\boldsymbol{\rho}) U(\boldsymbol{\rho}) \psi_{\mu}(\boldsymbol{\rho}) \right|^2 \delta(k_{\mu}^2 - k^2), \quad (20)$$

where ψ_0 is the “localized” solution of Eq. (1) (without evanescent leakage), ψ_{μ} and k_{μ} are the propagating eigenfunctions and eigenvalues of Eq. (1), respectively. It is seen from Eq. (20) that small additional leakage requires small matrix elements $\langle \psi_0 | U | \psi_{\mu} \rangle$. Since the functions ψ_{μ} belong to the continuous spectrum, it might seem that this requirement is impossible to meet. This is, however, not the case, since normalization factor of $|\psi_{\mu}|^2$ is the inverse system size. To demonstrate that additional leakage can be small, it is convenient to use the explicit form of ψ_{μ} , namely, the plane waves, and rewrite Eq. (20) in the form

$$\frac{1}{\tau_U} = \iint d\boldsymbol{\rho}_1 d\boldsymbol{\rho}_2 U(\boldsymbol{\rho}_1) S(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2) U(\boldsymbol{\rho}_2), \quad (21)$$

where the kernel $S(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2)$ is defined as

$$\begin{aligned} S(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2) &= \psi_0^*(\boldsymbol{\rho}_1) \psi_0(\boldsymbol{\rho}_2) \left(\frac{c}{k}\right) \int \frac{d\mathbf{q}}{2\pi} e^{iq(\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2)} \delta(q^2 - k^2) \\ &= \frac{c}{2k} \psi_0^*(\boldsymbol{\rho}_1) \psi_0(\boldsymbol{\rho}_2) J_0(k|\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2|), \end{aligned} \quad (22)$$

It is seen from Eq. (22) that the kernel $S(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2)$ is exponentially small if one of the points, $\boldsymbol{\rho}_1$ or $\boldsymbol{\rho}_2$, is located outside the region, occupied by the “body” of $\psi_0(\boldsymbol{\rho})$. When both $\boldsymbol{\rho}_1$ and $\boldsymbol{\rho}_2$ are located inside this region, then the characteristic spatial scale of the kernel, S , is $|\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2| \sim k^{-1}$. Thus, for the sake of our qualitative discussion, we can replace $S(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2)$ by $A^{-1} \theta(k^{-1} - |\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2|)$, where $A \sim |\psi_0(0)|^{-2}$ is the area of the fluctuation, and restrict integration in Eq. (21) to the region of the area A .

Averaging over disorder configurations in Eq. (21) yields the mean value of the additional leakage

$$\frac{1}{\tau_e} = \left(\frac{c}{k}\right) \int \frac{d\mathbf{q}}{2\pi} \left\langle \left| \int d\boldsymbol{\rho} \psi_0^*(\boldsymbol{\rho}) U(\boldsymbol{\rho}) e^{iq\boldsymbol{\rho}} \right|^2 \right\rangle \delta(q^2 - k^2) = \frac{c}{l_e}, \quad (23)$$

which is determined by *typical* values of $U(\boldsymbol{\rho})$. We note that l_e is of the order of the mean free path, l . The fact that $l_e \sim l$ can be seen from Eqs. (21)-(23), taking into account that the area A is always bigger than R_c^2 . Moreover, for short-range disorder, Eqs. (21)-(23) suggest that $l_e \approx l$.

In order to have τ_U^{-1} small, the actual value of $U(\boldsymbol{\rho})$ should be suppressed with respect to typical in *each* box of a size k^{-1} . Then the condition $\tau_U > \tau$ requires the suppression factor to exceed τ/τ_e . The corresponding probability can be estimated as

$$P(\tau) = (\tau_e/\tau)^N = (Q/\omega\tau_e)^{-N}, \quad (24)$$

where $N \sim k^2 A$ is the number of boxes that “cover” the area of localization of ψ_0 . In particular, for the ring-shaped fluctuations we have $N \sim w\rho_0 k^2$. We conclude that, due to additional leakage, the prefactor in the density of traps is *exponentially* small, i.e., $P(\tau) \sim \exp[-k^2 A \ln(Q/kl)]$. The exponent $k^2 A \ln Q$ should be compared with the main exponent, found in Sec. II. Since the above estimate assumed a short-range disorder, $kR_c \ll 1$, the principal exponent is $2(\pi^3/3)^{1/2} kl \ln Q$ [Eq. (15)]. Recall that, for short-range disorder we have $w \sim k^{-1}(k\rho_0/\ln Q)^{1/3}$, $\rho_0 \sim m/k$, so that the product $k^2 A$ is $\sim m^{4/3} \ln^{-1/3}(Q/kl)$. The final estimate for the exponent, originating from the prefactor, is $\sim m^{4/3} \ln^{2/3} Q$. This suggests that traps are the more frequent the smaller is the angular momentum, m . On the other hand, in derivation of the main exponent we have assumed that $m > \ln Q$. So that the minimal value of $(m^2 \ln Q)^{2/3}$ is $\sim \ln^2 Q$. This leads us to the conclusion that the main exponent *dominates* over the exponent, originating from the prefactor. This is because for the main exponent we have $kl \ln Q > \ln^2 Q$, since the relation $kl > \ln Q$ was assumed in the derivation of the main exponent.

The above qualitative analysis leading to Eq. (24) was restricted, for simplicity, to the case of the short-range disorder. In the next subsection we present a rigorous calculation of $P(\tau)$ for the more realistic case of a smooth disorder.

B. Derivation of $P(\tau)$

Denote by $p(\tau)$ the probability *density* that the lifetime with respect to additional leakage is equal to τ , so that $p(\tau) = dP(\tau)/d\tau$. The rigorous definition of this density reads

$$p(\tau) = \mathcal{N} \int \mathcal{D}\{U\} e^{-\mathcal{P}\{U\}} \delta(\tau - \tau_U), \quad (25)$$

where the normalization constant is defined as

$$\mathcal{N} = \left[\int \mathcal{D}\{U\} e^{-\mathcal{P}\{U\}} \right]^{-1} \quad (26)$$

and $\mathcal{P}\{U\}$ is given by

$$\mathcal{P}\{U\} = \frac{1}{2U_0^2} \iint d\boldsymbol{\rho}_1 d\boldsymbol{\rho}_2 U(\boldsymbol{\rho}_1) \kappa(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2) U(\boldsymbol{\rho}_2), \quad (27)$$

where $\kappa(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2)$ is related to the correlator $K(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2)$, defined by Eq. (4), as

$$\int d\boldsymbol{\rho}' \kappa(\boldsymbol{\rho}_1, \boldsymbol{\rho}') K(\boldsymbol{\rho}', \boldsymbol{\rho}_2) = \delta(\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2). \quad (28)$$

Since we are dealing with photons, the value U_0 can be expressed through the r.m.s. fluctuation of the dielectric constant as $U_0 = k_0^2 \Delta$ [see Eq. (3)]. Using the integral representation of the δ -function, Eq. (25) can be rewritten in the form

$$p(\tau) = \frac{\mathcal{N}}{2\pi} \int_{-\infty}^{\infty} dt e^{it/\tau} \int \mathcal{D}\{U\} e^{-\mathcal{P}_t\{U\}}, \quad (29)$$

where the auxiliary functional, $\mathcal{P}_t\{U\}$, is defined as

$$\mathcal{P}_t\{U\} = \mathcal{P}\{U\} + i \frac{t}{\tau_U} = \iint d\boldsymbol{\rho}_1 d\boldsymbol{\rho}_2 U(\boldsymbol{\rho}_1) \mathcal{K}_t(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2) U(\boldsymbol{\rho}_2). \quad (30)$$

The kernel, $\mathcal{K}_t(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2)$, of the functional $\mathcal{P}_t\{U\}$ has the form

$$\mathcal{K}_t(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2) = \frac{1}{2} k_0^{-4} \Delta^{-2} \kappa(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2) + it S(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2) \quad (31)$$

Following the standard procedure of functional integration, we present the fluctuation $U(\boldsymbol{\rho})$ as a linear combination

$$U(\boldsymbol{\rho}) = \sum_{\mu} C_{t,\mu} \phi_{t,\mu}(\boldsymbol{\rho}) \quad (32)$$

where $\phi_{t,\mu}(\boldsymbol{\rho})$ are the eigenfunctions of the operator $\hat{\mathcal{K}}_t$ with the kernel \mathcal{K}_t

$$\hat{\mathcal{K}}_t \phi_{t,\mu}(\boldsymbol{\rho}) = \int d\boldsymbol{\rho}_1 \mathcal{K}_t(\boldsymbol{\rho}, \boldsymbol{\rho}_1) \phi_{t,\mu}(\boldsymbol{\rho}_1) = \Lambda_{t,\mu} \phi_{t,\mu}(\boldsymbol{\rho}), \quad (33)$$

and $\Lambda_{t,\mu}$ are the corresponding eigenvalues.

At this point, we note that the operator $\hat{\mathcal{K}}_t$ is non-hermician. As a consequence, the functions $\phi_{t,\mu}$ form an orthogonal basis if the definition of the scalar product is modified to $\langle \phi_1 | \phi_2 \rangle = \int d\boldsymbol{\rho} \phi_1(\boldsymbol{\rho}) \phi_2(\boldsymbol{\rho})$. To see this, note that the kernel $\mathcal{K}_t(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2)$ is symmetric with respect to interchange $\boldsymbol{\rho}_1 \leftrightarrow \boldsymbol{\rho}_2$, $\mathcal{K}_t(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2) = \mathcal{K}_t(\boldsymbol{\rho}_2, \boldsymbol{\rho}_1)$. Then it follows from Eq. (33)

$$\begin{aligned} \iint d\boldsymbol{\rho}_1 d\boldsymbol{\rho}_2 \phi_{t,\mu_1}(\boldsymbol{\rho}_1) \mathcal{K}_t(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2) \phi_{t,\mu_2}(\boldsymbol{\rho}_2) &= \\ &= \int d\boldsymbol{\rho}_1 \phi_{t,\mu_1}(\boldsymbol{\rho}_1) \int d\boldsymbol{\rho}_2 \mathcal{K}_t(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2) \phi_{t,\mu_2}(\boldsymbol{\rho}_2) \\ &= \Lambda_{t,\mu_2} \int d\boldsymbol{\rho}_1 \phi_{t,\mu_1}(\boldsymbol{\rho}_1) \phi_{t,\mu_2}(\boldsymbol{\rho}_1). \end{aligned} \quad (34)$$

On the other hand, it also follows from Eq. (33)

$$\begin{aligned} \iint d\boldsymbol{\rho}_1 d\boldsymbol{\rho}_2 \phi_{t,\mu_1}(\boldsymbol{\rho}_1) \mathcal{K}_t(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2) \phi_{t,\mu_2}(\boldsymbol{\rho}_2) &= \\ &= \int d\boldsymbol{\rho}_2 \phi_{t,\mu_2}(\boldsymbol{\rho}_2) \int d\boldsymbol{\rho}_1 \mathcal{K}_t(\boldsymbol{\rho}_2, \boldsymbol{\rho}_1) \phi_{t,\mu_1}(\boldsymbol{\rho}_1) \\ &= \Lambda_{t,\mu_1} \int d\boldsymbol{\rho}_2 \phi_{t,\mu_1}(\boldsymbol{\rho}_2) \phi_{t,\mu_2}(\boldsymbol{\rho}_2). \end{aligned} \quad (35)$$

Comparing the r.h.s. of Eqs. (34) and (35), we have

$$\langle \phi_{t,\mu_1} | \phi_{t,\mu_2} \rangle = \int d\boldsymbol{\rho} \phi_{t,\mu_1}(\boldsymbol{\rho}) \phi_{t,\mu_2}(\boldsymbol{\rho}) = \delta_{\mu_1, \mu_2}. \quad (36)$$

Using the expansion Eq. (32), the functional integral Eq. (29) reduces to the integration over *all* complex coefficients $C_{t,\mu}$ in the expansion Eq. (32). However, due to the restriction that $U(\boldsymbol{\rho})$ is real, the pair $\{\text{Re}C_{t,\mu}, \text{Im}C_{t,\mu}\}$ can be “rotated” in such a way that instead of integrals over real and imaginary parts we get a single integral along the real axis.

$$p(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt e^{it/\tau} \frac{\int_{-\infty}^{\infty} \mathcal{D}\{\tilde{C}_{t,\mu}\} \exp\left(-\sum_{\mu} \Lambda_{t,\mu} \tilde{C}_{t,\mu}^2\right)}{\int_{-\infty}^{\infty} \mathcal{D}\{\tilde{C}_{0,\mu}\} \exp\left(-\sum_{\mu} \Lambda_{0,\mu} \tilde{C}_{0,\mu}^2\right)}. \quad (37)$$

To proceed further, we note that the real parts of all the eigenvalues $\Lambda_{t,\mu}$ are positive. Indeed, it follows from Eqs. (31) and (33) that

$$\text{Re}\Lambda_{t,\mu} = \left(\frac{1}{2k_0^4 \Delta^2}\right) \frac{\iint d\boldsymbol{\rho}_1 d\boldsymbol{\rho}_2 \phi_{t,\mu}^*(\boldsymbol{\rho}_1) \kappa(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2) \phi_{t,\mu}(\boldsymbol{\rho}_2)}{\iint d\boldsymbol{\rho} |\phi_{t,\mu}(\boldsymbol{\rho})|^2}. \quad (38)$$

Since κ is positively defined, $\text{Re}\Lambda_{t,\mu}$ is positive. The last remark ensures the convergence of all the gaussian integrals in Eq. (37). Thus, we obtain from Eq. (37)

$$p(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt e^{it/\tau} \left(\prod_{\mu} \frac{\Lambda_{0,\mu}}{\Lambda_{t,\mu}} \right)^{1/2}. \quad (39)$$

Since the product of the eigenvalues of an operator is equal to its determinant, Eq. (39) can be rewritten in the form

$$p(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt e^{it/\tau} \left(\frac{\det \hat{\mathcal{K}}_0}{\det \hat{\mathcal{K}}_t} \right)^{1/2}. \quad (40)$$

It is convenient to present the ratio of determinants in the integrand of Eq. (40) as a single determinant. This is achieved through the following sequence of steps

$$\begin{aligned} \frac{\det \hat{\mathcal{K}}_0}{\det \hat{\mathcal{K}}_t} &= \frac{1}{\det \hat{\mathcal{K}}_0^{-1} \det \hat{\mathcal{K}}_t} = \frac{1}{\det \left(\hat{\mathcal{K}}_0^{-1} \hat{\mathcal{K}}_t \right)} = \\ &= \frac{1}{\det \left[2k_0^4 \Delta^2 \hat{\mathcal{K}}^{-1} \left(\frac{1}{2} k_0^{-4} \Delta^{-2} \hat{\mathcal{K}} + it \hat{S} \right) \right]} = \\ &= \frac{1}{\det \left[1 + 2itk_0^4 \Delta^2 \hat{K} \hat{S} \right]}, \end{aligned} \quad (41)$$

where we have used the explicit form (31) of the operator $\hat{\mathcal{K}}_t$. The operators \hat{K} and \hat{S} are the integral operators with the kernels $S(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2)$ [Eq. (22)] and $K(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2)$, respectively. We recall that the operator \hat{K} is the inverse of the operator $\hat{\mathcal{K}}$ [Eq. (28)]. Upon transformation (41), the expression (40) for $p(\tau)$ takes the form

$$\begin{aligned} p(\tau) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dt \frac{e^{it/\tau}}{\sqrt{\det \left[1 + 2itk_0^4 \Delta^2 \hat{K} \hat{S} \right]}} \\ &= \frac{1}{\pi} \int_0^{\infty} dt \operatorname{Re} \left\{ \frac{e^{it/\tau}}{\sqrt{\det \left[1 + 2itk_0^4 \Delta^2 \hat{K} \hat{S} \right]}} \right\}. \end{aligned} \quad (42)$$

To proceed further, we need to analyze the properties of eigenfunctions and eigenvalues of the operators \hat{K} and \hat{S} .

1. Properties of operator \hat{K} .

It is easy to see that the eigenfunctions of \hat{K} are plane waves. Indeed, since the kernel K depends only on the difference $(\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2)$, we have

$$\hat{K} e^{ip\rho} = \int d\boldsymbol{\rho}_1 K(\boldsymbol{\rho} - \boldsymbol{\rho}_1) e^{ip\rho_1} = \tilde{K}(p) e^{ip\rho}, \quad (43)$$

so that eigenvalues of \hat{K} are the Fourier components of the correlator $K(\rho)$. Thus, these eigenvalues are strongly suppressed if $p > R_c^{-1}$. For the particular case of gaussian correlator we have

$$\tilde{K}(p) = \int d\boldsymbol{\rho} K(\rho) e^{ip\rho} = \pi R_c^2 e^{-p^2 R_c^2/4}. \quad (44)$$

2. Properties of operator \hat{S} .

Although the eigenfunctions, $\xi_\mu(\boldsymbol{\rho})$, of the operator \hat{S} are not plane waves, their width in the k -space is narrow (of the order of inverse spatial extent of the function ψ_0) as it can be seen from Eq. (22). To estimate the eigenvalues, λ_μ , of \hat{S} , defined as

$$\int d\boldsymbol{\rho}_1 S(\boldsymbol{\rho}, \boldsymbol{\rho}_1) \xi_\mu(\boldsymbol{\rho}_1) = \lambda_\mu \xi_\mu(\boldsymbol{\rho}), \quad (45)$$

we first note that these eigenvalues satisfy the following sum rule

$$\sum_\mu \lambda_\mu = \frac{c}{2k}. \quad (46)$$

This rule follows from the identity

$$\begin{aligned} \sum_\mu \lambda_\mu &= \iint d\boldsymbol{\rho} d\boldsymbol{\rho}_1 S(\boldsymbol{\rho}, \boldsymbol{\rho}_1) \sum_\mu \xi_\mu^*(\boldsymbol{\rho}) \xi_\mu(\boldsymbol{\rho}_1) \\ &= \int d\boldsymbol{\rho} S(\boldsymbol{\rho}, \boldsymbol{\rho}) = \frac{c}{2k} \int d\boldsymbol{\rho} |\psi_0(\boldsymbol{\rho})|^2, \end{aligned} \quad (47)$$

in which the completeness of the set $\xi_\mu(\boldsymbol{\rho})$ is used, so that

$$\sum_\mu \xi_\mu^*(\boldsymbol{\rho}) \xi_\mu(\boldsymbol{\rho}_1) = \delta(\boldsymbol{\rho} - \boldsymbol{\rho}_1). \quad (48)$$

The reason why Eq. (46) allows to estimate the eigenvalues is their specific distribution. Namely, the first $N \approx k^2 A$ eigenvalues are almost equal to each other, while the eigenvalues with numbers $\mu > N$ fall off rapidly, faster than $(N/\mu)^4$. This rapid fall off of λ_μ has a simple explanation. The eigenfunctions corresponding to large μ are close to plane waves, so that the value of λ_μ is determined by the integral of the rapidly oscillating plane wave over the area A and is small due to the cancellation. On the contrary, for $\mu < N$ the eigenfunction changes weakly within the area A . Thus, for $\mu < N$ we have $\lambda_\mu \approx \lambda_0 \sim c/(kN)$. For the particular case of the ring-shaped fluctuations $N \approx 2\pi\rho_0 w k^2$, so that $\lambda_0 \approx c/(\rho_0 w k^3)$.

3. Evaluation of the integral (42).

Short-range disorder. In this case $\tilde{K}(p) \approx \tilde{K}(0) \equiv \pi R_c^2$, so that the contribution to the integrand of Eq. (42) comes from N eigenvalues of the operator \hat{S} , i.e., $\det[1 + 2itk_0^4 \Delta^2 \hat{K} \hat{S}] \approx [1 + 2i\pi t k_0^4 \tilde{\Delta}^2 \lambda_0]^N$, where $R_c \Delta \rightarrow \tilde{\Delta}$ when $R_c \rightarrow 0$. Then the integration over t in (42) can be easily performed, yielding

$$\begin{aligned} p(\tau) &= \frac{1}{4\tau(N/2 - 1)!} \left(2\pi \tau \lambda_0 k_0^4 \tilde{\Delta}^2 \right)^{2-N/2} \exp \left[- \left(2\pi \tau \lambda_0 k_0^4 \tilde{\Delta}^2 \right)^{-1} \right] \\ &\sim \exp \left[-\frac{N}{2} \ln \left(\frac{Q}{kl_e} \right) - \left(\frac{N}{8} \right) \frac{kl_e}{Q} \right], \end{aligned} \quad (49)$$

where we have used the large- N asymptotics of $N!$ and the fact that $\lambda_0 \approx c/(kN)$. Since the first term in the exponent is much larger than the second one, we recover with exponential accuracy the form of $P(\tau)$, obtained in the qualitative consideration.

Smooth disorder. In the case of a smooth disorder, $kR_c > 1$, it is the fast decay of $\tilde{K}(p)$ [Eq. (44)] rather than λ_μ , that introduces a “cutoff” of the determinant in Eq. (42). In this case it is convenient to rewrite the determinant in Eq. (42) in the form

$$\begin{aligned} \det[1 + 2itk_0^4 \Delta^2 \hat{K} \hat{S}] &= \prod_\mu [1 + 2it\lambda_0 k_0^4 \Delta^2 \tilde{K}(p_\mu)] \\ &= \exp \left[\sum_\mu \ln \left(1 + 2it\lambda_0 k_0^4 \Delta^2 \tilde{K}(p_\mu) \right) \right]. \end{aligned} \quad (50)$$

The sum over μ in the exponent of (50) goes over both projections, p_x and p_y , of the momentum p . For ring-shaped fluctuations it is natural to consider the radial and azimuthal components of p . Since $w \sim R_c$ and $\rho_0 \gg R_c$ the contribution to the sum comes only from a single radial component, while the sum over angular component can be replaced by an integral. Thus we obtain

$$\det \left[1 + 2itk_0^4 \Delta^2 \hat{K} \hat{S} \right] = \exp \left[\rho_0 \int_0^\infty dp \ln \left(1 + 2it\lambda_0 k_0^4 \Delta^2 \tilde{K}(p) \right) \right]. \quad (51)$$

The main contribution to the integral (51) comes from the domain $\tau\lambda_0 k_0^4 \Delta^2 \tilde{K}(p) \sim 1$, so that $p > R_c^{-1}$. It is instructive to perform further calculations for more general form of $\tilde{K}(p)$, namely $\tilde{K}(p) \sim \exp[-(pR_c)^n]$, which reduces to Eq. (44) when $n = 2$. Substituting this form into Eq. (51) and performing two subsequent integration by parts, we get

$$\begin{aligned} \det \left[1 + 2itk_0^4 \Delta^2 \hat{K} \hat{S} \right] &= \exp \left[\frac{i\rho_0}{R_c} \left(\frac{n}{n+1} \right) \int_0^\infty dw w^{(n+1)/n} \frac{e^{w-w_t}}{(e^{w-w_t} + i)^2} \right] \\ &= \exp \left[\frac{i\rho_0}{R_c} \left(\frac{n}{n+1} \right) w_t^{(n+1)/n} \int_{-\infty}^\infty dw \frac{e^w}{(e^w + i)^2} \right], \end{aligned} \quad (52)$$

where $w_t = \ln(2\pi t\lambda_0 R_c^2 k_0^4 \Delta^2)$. In the second identity we made use of the fact that the function $\frac{e^{w-w_t}}{(e^{w-w_t} + i)^2}$ has a sharp maximum at $w = w_t$. The remaining integral in Eq. (52) can be evaluated exactly, $\int_{-\infty}^\infty dw \frac{e^w}{(e^w + i)^2} = -i$, so that Eq. (52) takes the form

$$\det \left[1 + 2itk_0^4 \Delta^2 \hat{K} \hat{S} \right] = \exp \left[\frac{\rho_0}{R_c} \left(\frac{n}{n+1} \right) w_t^{(n+1)/n} \right]. \quad (53)$$

Substituting this form into Eq. (42), we obtain

$$p(\tau) = \frac{1}{\pi} \int_0^\infty dt \cos(t/\tau) \exp \left[-\frac{n\rho_0}{2(n+1)R_c} \ln^{(n+1)/n} (2\pi t\lambda_0 R_c^2 k_0^4 \Delta^2) \right]. \quad (54)$$

Evaluating the above integral with an exponential accuracy yields

$$p(\tau) \sim \exp \left[-\frac{n\rho_0}{2(n+1)R_c} \ln^{(n+1)/n} (2\pi\tau\lambda_0 R_c^2 k_0^4 \Delta^2) \right]. \quad (55)$$

Since $p(\tau)$ and $P(\tau)$ have the same exponential dependence, the final expression for the prefactor, $P(\tau)$, takes the form

$$P(\tau) \sim \exp \left[-\frac{n\rho_0}{2(n+1)R_c} \ln^{(n+1)/n} (Q/kl_a) \right], \quad (56)$$

where τ in the argument of the logarithm was replaced by the trapping time $\omega^{-1}Q$. For particular case of the gaussian correlator (44) the probability that the lifetime with respect to additional leakage is longer than $\omega^{-1}Q$ is given by

$$P(\omega^{-1}Q) \sim \exp \left[-\frac{\rho_0}{3R_c} \ln^{3/2} (Q/kl_a) \right]. \quad (57)$$

Note, that for the short-range disorder, [Eq. (49)], the number N was the number of sections with the area $\sim k^{-2}$, which “cover” the ring-shaped trap. Correspondingly, for the smooth disorder the ratio ρ_0/R_c in the exponent is the number of squares with the side $\sim R_c$ that cover the trap. On the other hand, as it is seen from Eq. (56), the power of the logarithm is specific for the gaussian correlator.

C. Optimal Ring

Combining the main exponent for the gaussian correlator, Eq. (17), and the corresponding prefactor, Eq. (57), we obtain

$$\begin{aligned} \mathbf{N}_m(Q) &= P(\omega_c Q) e^{-S_m(Q)} \\ &\sim \exp \left[-\frac{m}{3kR_c} \ln^{3/2} Q - 3^{4/3} 4^{-5/3} \pi \frac{kl}{(kR_c)^2} m^{-1/3} \ln^{4/3} Q \right]. \end{aligned} \quad (58)$$

It is seen that the dependence of the main exponent and the prefactor on m are opposite. The prefactor, reflecting the “vulnerability” of the ring, falls off with m , while the main exponent favors large m value for which the Q -factor is higher. As a result of the competition between the two tendencies $\mathbf{N}_m(Q)$ has a sharp maximum at optimal value of m given by

$$m_{opt} = \frac{3\pi^{4/3}}{4^{5/4}} \left(\frac{l}{R_c} \right)^{3/4} \ln^{-1/8} Q \quad (59)$$

Substituting $m = m_{opt}$ into Eq. (58), we arrive at the final result

$$\mathbf{N}_{opt}(Q) = \exp \left[-2^{-1/2} \pi^{3/4} \frac{(kl)^{3/4}}{(kR_c)^{7/4}} \ln^{11/8} Q \right]. \quad (60)$$

If we use a general form of the prefactor Eq. (56), then the changes in Eq. (60) amount to an additional factor $2^{-13/4} (3n)^{1/4} (n+1)^{-5/4} (11n+2)$. Also the power of $\ln Q$ in Eq. (60) modifies to $(5n+1)/4n$. Overall, these changes are inessential, so that the result, Eq. (60), is rather robust. It shows how the trapping is enhanced due to a smooth disorder, when the additional leakage is taken into account. Without the prefactor this enhancement manifested itself through the combination $(kR_c)^2$ in the denominator of the main exponent, S_m . With the prefactor, $(kR_c)^2$ is replaced by $(kR_c)^{7/4}$, so that the enhancement is weaker, but insignificantly.

As a final remark, we note that additional leakage, caused by the scattering out of the plane, can be incorporated into the theory in a similar fashion as the in-plane additional leakage. Corresponding changes are outlined in the end of Sec.II. Recall also, that for the smooth disorder the suppression of $\mathbf{N}_m(Q)$ due to additional leakage is dominated by the in-plane scattering processes.

IV. CONCLUSION

In the present paper we studied a new type of solutions [66,76] of the wave equation, Eq. (1), in a weakly disordered medium. The solutions, dubbed as “almost localized” states, describe a wave which is confined primarily to a small ring. In an *open* sample, of size L much smaller than the two-dimensional localization length ξ , the almost localized states correspond to sharp resonances, residing in the high- Q ring-shaped cavities, as discussed throughout the paper. However, in a *closed* sample -which would require perfectly reflecting walls- the resonances turn into true eigenstates, whose almost entire weight is located at the rings. In this respect the “almost localized” states differ from the “prelocalized” states, extensively studied in the context of electronic transport [72-75].

We have provided a quantitative theory of the almost localized states and the associated random resonators, and pointed out their relevance for the phenomenon of random lasing. We stress, however, that these random resonators exist already in the *passive* medium, and gain is only needed “to make them visible”. Moreover, the resonators are “self-formed”, in the sense that no sharp features (like Mie scatterers or other “resonant entities”) are introduced: the model is defined by Eq.(1), which describes a featureless dielectric medium with fluctuating dielectric constant.

Our study substantiates the intuitive image [43,44,58] of a resonant cavity as a closed-loop trajectory of a light wave bouncing between the point-like scatterers. The intuitive picture in [43,44,58] assumed that light can propagate along a loop of scatterers by simply being scattered from one scatterer to another. Such a picture, however, is unrealistic due to the scattering out of the loop. We have demonstrated that the scenario of light traveling along closed loops can be remedied. In our picture the “loops”, i.e., the random resonators, can be envisaged as rings with dielectric constant larger than the average value. The reason why such rings are able to trap the light is that the constituting scatterers act as a *single entity*: only the coherent multiple scattering of light by *all* the scatterers in the resonator can provide trapping. We have also established that correlations in the fluctuating part of the dielectric constant highly facilitate trapping.

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